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Innovation The Research Concept

Infinitude Primes with Number Theoretic Function-σ

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Abstract

This research paper, deals with the nature of the numbers belongs to the collection of the integers $C_n = \{N_n[\sigma(n)] = n\,\sigma(n) + 1$, where $n \in N$, and $\sigma(n)$represents the sum of positive divisors of $n\}$. Here, we consider only three cases- when $n = p$, $n = 2.p$, and $n = 3.p$. We observe that the collection $C_n$contains infinitude primes of the form $N_n[\sigma(n)]$ in all three cases . Also, in this paper we calculate the value $\sigma(N_n[\sigma(n)])$, for all three cases and find nice results regarding Möbious function μ

Keywords: Composite Number, Möbious Function, Number Theoretic Function, Prime Number.

Introduction

Our discussion starts from the theorem [3]-For every $n \geq 2$, $n \in N \varphi(n).\ \psi(n).\ \sigma(n) \geq n^3 + n^2 - n - 1$.

Let us consider the $n \geq 2$ [1, 2,3]:

$n = \prod_{i=1}^{r} p_i^{\alpha_i}$,

where $r \geq 1$, and each α_i's≥ 1 are natural numbers and p_i's are distinct primes.

The number theoretic function σ defined as [1, 3]-

$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_{i+1}} - 1}{p_i - 1}$, and $\sigma(1) = 1$.

And, the Möbious function defined as [1, 4]-

$\mu(n) = 1,\ if\ n = 1$

$\mu(n) = 0,\ if\ p^2 | n\ where\ p\ is\ a\ prime$, and

$\mu(n) = (-1)^r,\ if\ n = \prod_{i-1}^{r} p_i,\ where\ each\ p_i's\ are\ ptime\ number$

There are several research papers [1,2,3, 4, 5, 6] in which characterises the number theoretic functions are discussed. In 2010, M. Vassilev- Missana [5], and in 2011, M. Vassilev- Missana, P. Vassilev [6] established some new results on multiplicative functions with strictly positive values. These results motivate to us to construct the collection of the integers $C_n = \{N_n[\sigma(n)] = n\,\sigma(n) + 1$, where $n \in N$, and $\sigma(n)$represents the sum of positive divisors of $n\}$. Consider

$N_n[\sigma(n)] = n\sigma(n) + 1$ (1)

where $n \in N$, and $\sigma(n)$represents the sum of positive divisors of n.

Objective of the Study

The general objective of our research papers is to investigate and justify that the collection of integers

$C_n = \{N_n[\sigma(n)] = n\sigma(n) + 1,\ where\ n = p,\ n = 2p$, and $n = 3p\}$

admits infinitude primes

Study Duration: From June 2021 to till now.

Hypothesis

The main hypothesis of our research paper is that "There are infinitely many primes of the form $N_n[\sigma(n)] = n\sigma(n) + 1$ either n is composite or prime."

Analysis

Case-I: Here, we study the nature of the numbers for prime values of n, i.e. when $n = p$, p is prime.

Set $n = p$ in (1), we have-

$N_p[\sigma(p)] = p\sigma(p) + 1$

$N_p[\sigma(p)] = p(p + 1) + 1 = \sigma(p^2)$ (2)

Theorem 1. If $p \equiv 2\ (mod\ 4)$, then $N_p[\sigma(p)] \equiv 3\ (mod\ 4)$.

Proof: Since we know that there is only one prime $p = 2$ for which $p \equiv 2\ (mod\ 4)$. Therefore, from (2) we have
$N_2[\sigma(2)] = 7$
Clearly,
$N_p[\sigma(p)] \equiv 3\ (mod\ 4)$.

Theorem 2. If $p \equiv 1\ (mod\ 4)$, then $N_p[\sigma(p)] \equiv 3\ (mod\ 4)$.

Proof: If $p \equiv 1\ (mod\ 4)$, then $p^2 \equiv 1\ (mod\ 4)$,
Therefore, from (2) we have
$N_p[\sigma(p)] = 1 + 1 + 1 \equiv 3\ (mod\ 4)$
Clearly,
$N_p[\sigma(p)] \equiv 3\ (mod\ 4)$.

Theorem 3. If $p \equiv 3\ (mod\ 4)$, then $N_p[\sigma(p)] \equiv 1\ (mod\ 4)$.

Proof: If $p \equiv 3\ (mod\ 4)$, then $p^2 \equiv 1 (mod\ 4)$,
Therefore, from (2) we have
$N_p[\sigma(p)] = 3 + 1 + 1 \equiv 1\ (mod\ 4)$
Clearly,
$N_p[\sigma(p)] \equiv 1\ (mod\ 4)$.

The following table gives us the list of numbers $N_p[\sigma(p)]$ for $p \geq 2$.

Table-1
(All entries in red colour represents prime values)

S. No.	$n = p$	$N_n[\sigma(n)]$
1.	2	7
2.	3	13
3.	5	31
4.	7	$57 = 3\times19$
5.	11	$133 = 7\times19$
6.	13	$183 = 3\times61$
7.	17	307
8.	19	$381 = 3\times127$
9.	23	$553 = 7\times79$
10.	29	$871 = 13\times67$
11.	31	$993 = 3\times331$
12.	37	$1407 = 3\times7\times67$
13.	41	1723
14.	43	$1893 = 3\times631$
15.	47	$2257 = 37\times61$
16.	53	$2863 = 7\times409$
17.	59	3541
18.	61	$3783 = 3\times13\times97$

19.	67	$4557 = 3\times7^2\times31$
20.	71	5113
21.	73	$5403 = 3\times1801$
22.	79	$6321 = 3\times7^2\times43$
23.	83	$6973 = 19\times367$
24.	89	8011
25.	97	$9507 = 3\times3169$
26.	101	10303
27.	103	$10713 = 3\times3571$
28.	107	$11557 = 7\times13\times127$
29.	109	$11991 = 3\times7\times571$
30.	113	$12883 = 13\times991$
31.	127	$16257 = 3\times5419$
32.	131	17293
33.	137	$18907 = 3\times37\times73$
34.	139	$19461 = 3\times 13 \times499$
35.	149	$22351 = 7\times31\times103$
36.	151	$22953 = 3\times7\times1093$
37.	157	$24807 = 3\times8269$
38.	163	$26773 = 41\times653$
39.	167	28057
40.	173	30103
41.	179	$32221 = 7\times4603$
42.	181	$32943 = 3\times79\times139$
43.	191	$36673 = 7\times13^2\times31$
44.	193	$37443 = 3\times7\times1783$
45.	197	$39007 = 19\times2053$
46.	199	$39801 = 3\times13267$
47.	211	$44733 = 3\times13\times31\times37$
48.	223	$49953 = 3\times16651$
49.	227	$51757 = 73 \times709$
50.	229	$52671 = 3 \times97\times 681$

51.	233	$54523 = 7\times 7789$
52.	239	$57361 = 19\times3019$
53.	241	$58323 = 3\times19441$
54.	251	$63253 = 43\times1471$
55.	257	$66307 = 61\times1087$
56.	263	$69433 = 7^2\times13\times109$
57.	269	$72631 = 13\times37\times151$
58.	271	$73713 = 3\times24571$
59.	277	$77007 = 3\times7\times19\times193$
60.	281	$79243 = 109\times727$
61.	283	$80373 = 3\times73\times367$
62.	293	86143
63.	307	$94557 = 3\times43\times733$
64.	311	$97033 = 19\times5107$
65.	313	$98283 = 3\times181^2$
66.	317	$1893 = 7\times14401$
67.	331	$2257 = 3\times7\times5233$
68.	337	$2863 = 3\times43\times883$
69.	347	$120757 = 7\times13\times1327$
70.	349	$122151 = 3\times19\times2143$
.	.	.
.	1709	2922391

In consequence of theorem (1), (2), (3) and table- 1 indicates that there are there are infinitude primes of the form $N_n[\sigma(n)]$, when n is a prime number. Clearly, the collection $C_n = \{N_n[\sigma(n)],\ n = p\}$ contains infinitude primes.

Case- II: Here, we study the nature of the numbers for composite numbers $n = p.q$.

Let $n = p.\ q$ in (1), we have-

$N_{p.q}[\sigma(p, q)] = p.q\ \sigma(p.\ q) + 1$

$N_{p.q}[\sigma(p.q)] = p.q\ (p + 1)(q + 1) + 1$

$= [\sigma(p^2) - 1].\ [\sigma(q^2) - 1] + 1$ (3)

From (2), we have

$N_{p.q}[\sigma(p.q)] = \{N_p[\sigma(p)] - 1\}.\ \{N_q[\sigma(q)] - 1\} + 1$ (4)

Theorem 4. If $p\equiv2\ (mod\ 4)$, and $q\equiv1\ (mod\ 4)$ then $N_{p.q}[\sigma(p.q)] \equiv1\ (mod\ 4)$.

Proof: From theorem (1), and (2) we have

$N_p[\sigma(p)] - 1 \equiv2\ (mod\ 4)$, and

$N_q[\sigma(q)] - 1\equiv2\ (mod\ 4)$

Clearly,

$N_p[\sigma(p)] \equiv1\ (mod\ 4)$.

Theorem 5. If $p \equiv 2\ (mod\ 4)$, and $q \equiv 3\ (mod\ 4)$ then $N_{p.q}[\sigma(p.q)] \equiv 3\ (mod\ 4)$

Proof: From theorem (1), and (3) we have

$N_p[\sigma(p)] - 1 \equiv 2\ (mod\ 4)$, and

$N_q[\sigma(q)] - 1 \equiv 0\ (mod\ 4)$

Clearly,

$N_p[\sigma(p)] \equiv 1\ (mod\ 4)$.

Now set $q = 2$ and $p \equiv 1,\ 3\ (mod\ 4)$ we find $N_{2.p}[\sigma(2.p)] = 2.p.\sigma(2.p) + 1 = 6p.(p + 1) + 1$

The following table gives us the list of numbers $N_{2.p}[\sigma(2.p)]$ for $p \equiv 1,\ 3\ (mod\ 4)$ where $p \neq 2$.

Table- 2

(All entries in red colour represents prime values)

S. No.	$n = 2.p$	$N_n[\sigma(n)]$
1.	6	73
2.	10	181
3.	14	337
4.	22	$793 = 13\times61$
5.	26	1093
6.	34	$1837 = 11\times167$
7.	38	2281
8.	46	3313
9.	58	$5221 = 23\times227$
10.	62	5953
11.	74	$8437 = 11\times13\times59$
12.	82	10333
13.	86	11353
14.	94	13537
15.	106	$17173 = 13\times1321$
16.	118	$21241 = 11\times1931$
17.	122	$22693 = 11\times2063$
18.	134	27337
19.	142	$30673 = 37\times829$
20.	146	32413
21.	158	$28441 = 7\times17\times239$
22.	166	$41833 = 11\times3803$
23.	178	$48061 = 13\times3697$

24.	194	57037
25.	202	61813
26.	206	$64273 = 11\times5843$
27.	214	69337
28.	218	71287
29.	226	$77293 = 37\times2089$
30.	254	$97537 = 11\times8867$
31.	262	$103753 = 13\times23\times347$
32.	274	$112615 = 5\times101\times223$
33.	278	$116761 = 59\times1979$
34.	298	$134101 = 11\times73\times167$
35.	302	137713
36.	314	$147895 = 5\times11\times2689$
37.	326	$160393 = 107\times1499$
38.	334	$168337 = 13\times23\times563$
39.	346	$180613 = 109\times1657$
40.	358	$193321 = 97\times1993$
41.	362	$197653 = 239\times827$
42.	382	$220033 = 11\times83\times241$
43.	386	$224653 = 11\times13\times1571$
44.	394	$234037 = 227\times1031$
45.	398	238801
46.	422	$267127 = 7\times31\times1231$
47.	446	$299713 = 23\times83\times157$
48.	454	$310537 = 193\times1609$
49.	458	$316021 = 71\times4451$
50.	466	327133
51.	478	344161
52.	482	349933
53.	502	379513
54.	514	$397837 = 11\times59\times613$

55.	526	416593
56.	538	$435781 = 23\times18947$
57.	542	$442273 = 13\times13\times2617$
58.	554	$462037 = 131\times3527$
59.	562	$475453 = 11\times43223$
60.	566	482233
61.	586	$516853 = 37\times61\times229$
62.	614	$567337 = 47\times12071$
63.	622	$582193 = 5771009$
64.	626	$589693 = 13\times45361$
65.	634	$602935 = 5\times120587$
66.	662	659353
67.	674	683437
68.	694	$724537 = 11\times65867$
69.	698	$732901 = 13\times56377$
.	.	.
.	.	.
.	3418	$17534341 = 11\times1594031$
.	.	.
.	15802	374602213

In the wake of theorem (4), (5), and table- 2 indicates that there are there are infinitude primes of the form $N_n[\sigma(n)]$, when n is composite number of the form $n = p.q$ with $q = 2$ and $p\equiv1,\ 3\ (mod\ 4)$excluding $p = 2$. Obviously, we observe that the collection $C_n = \{N_n[\sigma(n)],\ n = 2.p\}$ contains infinitude primes like case- I.

Case- III: Recall the theorems of Case- II, and set $q = 3$ and $p\equiv1,\ 2,\ \&\ 3\ (mod\ 4)$excluding $p = 3$, we find
$N_{2.p}[\sigma(3.p)] = 3.p.\sigma\,(3.p) + 1 = 12\,p.(p + 1) + 1$

The following table gives us the list of numbers $N_{3.p}[\sigma(3.p)]$ for $p\equiv1,\ 2,\ \&\ 3\ (mod\ 4)$ where $p\neq3$.

Table- 3
(All entries in red colour represents prime values)

S. No.	$n = 3.p$	$N_n[\sigma(n)]$
1.	6	73
2.	15	$361 = 19\times19$
3.	21	673
4.	33	$1585 = 5\times317$

5.	39	$2185 = 5\times19\times23$
6.	51	3673
7.	57	4561
8.	69	$6625 = 5^3\times53$
9.	87	$10441 = 53\times159$
10.	93	$11905 = 5\times2381$
11.	111	$16873 = 47\times359$
12.	121	$20665 = 5\times4133$
13.	129	$22705 = 5\times19\times241$
14.	141	27073
15.	159	$34345 = 5\times6869$
16.	177	$42481 = 23\times1847$
17.	183	$45385 = 5\times29\times313$
18.	201	54673
19.	213	$61345 = 5\times12269$
20.	219	$64825 = 5^2\times2593$
21.	237	$75841 = 149\times509$
22.	249	$83665 = 5\times29\times577$
23.	267	$96121 = 19\times5059$
24.	291	114073
25.	303	$123625 = 5^3\times23\times43$
26.	309	$128545 = 5\times47\times547$
27.	321	$138673 = 101\times1373$
28.	327	143881
29.	339	$154585 = 5\times43\times719$
30.	381	$195073 = 19\times10267$
31.	393	$207505 = 5\times47\times883$
32.	411	$226873 = 307\times739$
33.	417	$233521 = 293\times797$
34.	447	$268201 = 67\times4003$
35.	453	$275425 = 5^2\times23\times479$
36.	471	$297673 = 19\times15667$
37.	489	$320785 = 5\times64157$
38.	501	$336673 = 197\times1709$

39.	519	$361225 = 5^2 \times 14449$
40.	537	386641
41.	543	$395305 = 5\times173\times457$
42.	573	$440065 = 5\times283\times311$
43.	579	$449305 = 5\times23\times3907$
44.	591	$468073 = 23\times47\times433$
45.	597	$477601 = 29\times43\times383$
46.	633	$536785 = 5\times107357$
47.	669	$599425 = 5^2\times23977$
48.	681	$621073 = 167\times3719$
49.	687	632041
50.	699	$654265 = 19\times71\times97$
51.	717	$688321 = 23\times29927$
52.	723	$699865 = 5\times19\times53\times139$
53.	753	$759025 = 5^2\times97\times313$
54.	771	$795673 = 29\times27437$
55.	789	$833185 = 5\times71\times2347$
56.	807	$871531 = 163\times5347$
57.	813	$884545 = 5\times19\times9311$
58.	831	924073
59.	843	$950905 = 5\times190181$
60.	849	$964465 = 5\times67\times2879$
61.	879	$1033705 = 5\times29\times7129$
62.	921	1134673
63.	933	$1164385 = 5\times232877$
64.	939	$1179385 = 5\times235877$
65.	951	$1209673 = 19\times63667$
66.	993	$1318705 = 5\times23\times11467$
67.	1011	$1366873 = 173\times7901$
68.	1041	$1449073 = 19\times53\times1439$
69.	1047	1465801
.	5127	35068681
.	23703	$749204425 = 5^2\times71\times422087$

In the wake of theorem (4), (5), and table- 3 indicates that there are there are infinitude primes of the form $N_n[\sigma(n)]$, when n is composite number of the form

$n = p.q$ with $q = 3$ and $p\equiv 1, 2, \& 3 \ (mod\ 4)$excluding $p = 3$. Noted that the collection $C_n = \{N_n[\sigma(n)],\ n = 3.p\}$ contains infinitude primes like case- I & case- II.

Result and Discussion

Result 1- If $N_n[\sigma(n)]$ is a prime of the form $4k + 1$, then $\sigma\,(N_n[\sigma(n)]) = 2.\ m$, where $m\in N$ and $gcd\,(2,\ m) = 1$.

Result 2- If $\sigma\,(N_n[\sigma(n)]) = 2.\ m$, where $m\in N$ and $gcd\,(2,\ m) = 1$, then the value of Möbious function may or may not be equal to zero, i.e. either $\mu\{\sigma\,(N_n[\sigma(n)])\} = \pm 1$ or 0.

Result 3- If $N_n[\sigma(n)]$ is a prime of the form $4k + 3$, then $\sigma\,(N_n[\sigma(n)]) = 2^r.\ m$, where $r,\ m\in N,\ r\geq 2$ and $gcd\,(2,\ m) = 1$.

Result 4- If $N_n[\sigma(n)]$ is a prime of the form $4k + 3$, then the value of Möbious function must be zero, i.e. $\mu\{\sigma\,(N_n[\sigma(n)])\} = 0$.

Result 5- If $N_n[\sigma(n)]$ is a composite number with canonical representation $\prod_{i=1}^{r} p_i^{\alpha_i}$, where at least two $\alpha_i^{'}s$ are such as $\alpha_i \not\equiv 0\ (mod\ 2)$, then $\sigma\,(N_n[\sigma(n)]) = 2^r.\ m$, where $r,\ m\in N,\ r\geq 2$ and $gcd\,(2,\ m) = 1$.

Result 6- If $N_n[\sigma(n)]$ is a composite number such that $\sigma\,(N_n[\sigma(n)]) = 2^r.\ m$, where $r,\ m\in N,\ r\geq 2$ and $gcd\,(2,\ m) = 1$, then the value of Möbious function must be zero, i.e. $\mu\{\sigma\,(N_n[\sigma(n)])\} = 0$.

Conclusion

Eventually from the theorem 1,2,3 and 4 also consider table 1, 2, 3 we conclude that the collection of integers $C_n = \{N_n[\sigma(n)] = n\sigma(n) + 1,\ where\ n = p,\ n = 2p$, and $n = 3p\}$ admits infinitude primes of the form $N_n[\sigma(n)]$. Also, from results 1 to 6 we observe that if $N_n[\sigma(n)]$ is either a prime p of the form $4k + 3$ or composite number with canonical representation $\prod_{i=1}^{r} p_i^{\alpha_i}$, where at least two $\alpha_i^{'}s$ are such as $\alpha_i \not\equiv 0\ (mod\ 2)$, then the value of Möbious function must be zero, i.e. $\mu\{\sigma\,(N_n[\sigma(n)])\} = 0$, and when $N_n[\sigma(n)]$ is a prime of the form $4k + 1$, then the value of Möbious function may or may not be equal to zero, i.e. either $\mu\{\sigma\,(N_n[\sigma(n)])\} = \pm 1$ or 0.

Suggestions for the future Study

There are some open problems invites researchers-

1. What is the primality of the numbers of the form $N_n[\sigma(n)] = n\sigma(n) + 1$, where n belongs to the set of twin primes.
2. What is the primality of the numbers of the form $N_n[\sigma(n)] = n\sigma(n) + 1$, where n belongs to the set of Mersenne's numbers, Fermat's numbers, etc.

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